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LOWER BOUNDS OF THE GAP BETWEEN THE FIRST AND SECOND EIGENVALUES OF THE SCHRÖDINGER OPERATOR

BY

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ABSTRACT. In this paper the authors prove the following theorem:
Let Ω be a smooth strictly convex bounded domain in R^n and $V\colon \Omega \to R$ a nonnegative convex function. Suppose λ_1 and λ_2 are the first and second nonzero eigenvalues of the equation

$$-\Delta f + Vf = \lambda f, \qquad f|_{\partial\Omega} \equiv 0.$$

Then $\lambda_2 - \lambda_1 \geq \pi^2/d^2$, where d is the diameter of Ω.

Let $\Omega \subset R^n$ be a smooth strictly convex bounded domain and $W\colon \Omega \to R$ a nonnegative convex smooth function. The eigenvalues of the equation

$$(1) \qquad -\Delta f + Wf = \lambda f, \qquad f = 0, \quad \text{on } \partial\Omega$$

can be arranged in nondecreasing order as follows:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots.$$

B. Wong, S.-T. Yau and Stephen S.-T. Yau [**3**] proved that

$$(2) \qquad \lambda_2 - \lambda_1 \geq \pi^2/4d^2,$$

where d is the diameter of Ω. In this paper the authors will use the method of [**3** and **4**] to prove the following theorem:

THEOREM. *Let Ω be a smooth strictly convex bounded domain in R^n and $W\colon \Omega \to R$ a nonnegative convex function. Suppose λ_1 and λ_2 are the first and second nonzero eigenvalues of* (1). *Then*

$$(3) \qquad \lambda_2 - \lambda_1 \geq \pi^2/d^2,$$

where d is the diameter of Ω.

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In this paper the assumptions of all the lemmas are the same as those of the Theorem. We will not state them again.

Let f_1 and f_2 be the first and second eigenfunctions of (1); then $f_1(x) > 0$, $x \in \Omega$ [**2**], and $u = f_2/f_1$ is smooth to the boundary of Ω [**3**]. Suppose that

$$A = \max_{x\in\overline{\Omega}} u(x); \qquad -k = \min_{x\in\overline{\Omega}} u(x).$$

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We may assume that $A \geq k$, otherwise, we can use $-f_2$ instead of f_2.

Since $\int_\Omega f_1 f_2 = 0$ and $f_1 > 0$, $k > 0$. Setting $\tilde{u} = u/A$, we have $1 \geq \tilde{u} \geq -k/A = -\tilde{k}$ and $1 \geq \tilde{k} > 0$;

$$v = \left(\tilde{u} - \frac{1-\tilde{k}}{2}\right) \Big/ \left(\frac{1+\tilde{k}}{2}\right) \tag{4}$$

and

$$a = \frac{1-\tilde{k}}{1+\tilde{k}}; \qquad 1 > a \geq 0. \tag{5}$$

Then v is a smooth function on $\overline{\Omega}$. By computing, we have

$$\Delta v = -\lambda(v+a) - 2(\nabla v \cdot \nabla \log f_1), \tag{6}$$

where λ is $\lambda_2 - \lambda_1 > 0$.

LEMMA 1. *Let $z(v)$ be a smooth function defined on $\overline{\Omega}$ and $m > 0$ a constant. Suppose that*

$$G(x) = m|\nabla v|^2 \neq z(v), \tag{7}$$

$P \in \partial\Omega$ and $G(P) = \max_{x \in \overline{\Omega}} G(x)$. Then $\nabla v(P) = 0$.

PROOF. We can choose an orthonormal frame $l_1, l_2, \ldots, l_n$ around P such that l_1 is perpendicular to $\partial\Omega$ and pointing outward. We also use the notation $\partial/\partial x_1$ to denote the restriction of l_1 on $\partial\Omega$. Since $G(P)$ is the maximum of $G(x)$,

$$0 \leq \frac{\partial G}{\partial x_1}(P) = 2m \sum_{i=1}^{n} v_i v_{i1} + z' v_1. \tag{8}$$

Furthermore, in $\Delta v = -\lambda(v+a) - 2(\nabla v \cdot \nabla \log f_1)$, v and Δv are all smooth on $\overline{\Omega}$; hence

$$\nabla v \cdot \nabla \log f_1 = \frac{1}{f_1}\left[v_1 (f_1)_1 + \sum_{i=2}^{n} v_i (f_1)_i\right]$$

achieves finite value on $\partial\Omega$. But $f_1|_{\partial\Omega} \equiv 0$, thus

$$\left[v_1 (f_1)_1 + \sum_{i=2}^{n} v_i (f_1)_i\right]\Bigg|_{\partial\Omega} \equiv 0.$$

Since $f_1 \equiv 0$ and $\partial\Omega$ and l_i, $2 \leq i \leq n$, are the tangent vectors of $\partial\Omega$, $(f_1)_i|_{\partial\Omega} \equiv 0$, $2 \leq i \leq n$. Hence,

$$v_1 (f_1)_1 \equiv 0 \quad \text{on } \partial\Omega.$$

By Hopf's lemma, $\partial f_1/\partial x_1 \neq 0$. Therefore,

$$v_1 \equiv 0 \quad \text{on } \partial\Omega. \tag{9}$$

Putting (9) into (8), we have

$$0 \leq m \sum_{i=2}^{n} v_i v_{i1}(P) = \frac{\partial G}{\partial x_1}(P). \tag{10}$$

From the definition of the second fundamental form in R^n, we have (note $v_1 = 0$)

$$v_{1i} = -\sum_{j=2}^{n} h_{ij} v_j, \tag{11}$$

where (h_{ij}) is the second fundamental form. Putting (11) into (10), we obtain

$$0 \le -\sum_{i,j=2}^{n} m h_{ij} v_i v_j (P).$$

Since Ω is strictly convex, (h_{ij}) is positive defifnite; thus

$$0 \le -m \sum_{i,j=2}^{n} h_{ij} v_i v_j (P) \le 0.$$

Hence, $v_i(P) = 0$, $2 \le i \le n$, i.e., $\nabla v(P) = 0$.

LEMMA 2. *For any given* $b > 1$,

$$\frac{|\nabla v|^2}{b^2 - v^2} \le \lambda(1 + a).$$

PROOF. For $\varepsilon > 0$, consider the function defined on $\overline{\Omega}$

$$G(x) = |\nabla v|^2 + \lambda(1 + \varepsilon + a)v^2.$$

Suppose $G(P) = \max_{x \in \overline{\Omega}} G(x)$. If $P \in \partial\Omega$, by Lemma 1, we have $\nabla v(P) = 0$ and

$$G(x) \le G(P) = \lambda(1 + \varepsilon + a)v^2(P) \le \lambda(1 + \varepsilon + a). \tag{12}$$

Now we assume that $P \in \Omega$, and from the maximum principle we have that at P

$$0 = \nabla|\nabla v|^2 + \lambda(1 + \varepsilon + a)\nabla v^2$$

i.e.,

$$v_i v_{ij} = -\lambda(1 + \varepsilon + a) v v_j, \qquad 1 \le j \le n. \tag{13}$$

Also at P

(14)

$$\begin{aligned} 0 \ge \frac{1}{2}\Delta G(P) &= \sum_{i,j} v_{ij}^2 + \nabla v \cdot \nabla(\Delta v) + \lambda(1 + \varepsilon + a)v\Delta v + \lambda(1 + \varepsilon + a)|\nabla v|^2 \\ &= \sum_{i,j} v_{ij}^2 - |\nabla v|^2 - 2\nabla v \cdot \nabla(\nabla v \cdot \nabla \log f_1) - \lambda^2(1 + \varepsilon + a)v(v + a) \\ &\quad - 2\lambda(1 + \varepsilon + a)v(\nabla v \cdot \nabla \log f_1) + \lambda(1 + \varepsilon + a)|\nabla v|^2. \end{aligned}$$

If $\nabla v(P) = 0$, then (12) is valid. If $\nabla v(P) \ne 0$, we can choose an orthonormal frame such that $v_i(P) = 0$, $2 \le i \le n$, and $v_1(P) \ne 0$. (13) gives

$$v_{11} = -\lambda(1 + \varepsilon + a)v, \qquad v_{1i} = 0, \quad 2 \le i \le n. \tag{15}$$

Putting (15) into (14), we obtain

$$\begin{aligned}
0 &\geq v_{11}^2 + (\varepsilon + a)\lambda|\nabla v|^2 - 2v_1(\nabla v \cdot \nabla \log f_1)_1 - \lambda^2(1+\varepsilon+a)v^2 \\
&\quad - \lambda^2(1+\varepsilon+a)av - 2\lambda(1+\varepsilon+a)v(\nabla v \cdot \nabla \log f_1) \\
&= \lambda^2(1+\varepsilon+a)^2 v^2 - \lambda^2(1+\varepsilon+a)v^2 + (\varepsilon+a)\lambda|\nabla v|^2 - \lambda^2(1+\varepsilon+a)av \\
&\quad - 2\sum_{i=1}^n v_1 v_{i1}(\log f_1)_i - 2\sum_{i=1}^n v_1 v_i (\log f_1)_{i1} \\
&\quad - 2\lambda(1+\varepsilon+a)vv_1(\log f_1)_1 \\
&= \lambda(\varepsilon+a)[|\nabla v|^2 + \lambda(1+\varepsilon+a)v^2] - \lambda^2(1+\varepsilon+a)av - 2v_1v_{11}(\log f_1)_1 \\
&\quad - 2v^2(\log f_1)_{11} - 2(1+\varepsilon+a)vv_1(\log f_1)_1 \\
&= \lambda(\varepsilon+a)G(P) - \lambda^2(1+\varepsilon+a)av + 2v_1\lambda(1+\varepsilon+a)v(\log f_1)_1 \\
&\quad - 2v_1^2(\log f_1)_{11} - 2\lambda(1+\varepsilon+a)vv_1(\log f_1)_1.
\end{aligned}$$

Hence

$$0 \geq \lambda(\varepsilon+a)G(P) - \lambda^2(1+\varepsilon+a)av - 2v_1^2(\log f_1)_{11}. \tag{16}$$

Since W and Ω are all convex, $\log f_1$ is concave [**1**], in particular, $-(\log f_1)_{11}(P) \geq 0$. Noting that $v \leq 1$, we have

$$\lambda(1+\varepsilon+a)a \geq (\varepsilon+a)G(P), \quad G(x) \leq \lambda(1+\varepsilon+1)\frac{a}{\varepsilon+a} \leq \lambda(1+\varepsilon+a). \tag{17}$$

From (12) and (17) we can obtain that $G(x) \leq (1+\varepsilon+a)\lambda$, $x \in \overline{\Omega}$. This is

$$|\nabla v|^2 \leq \lambda(1+\varepsilon+a)(1-v^2) \leq \lambda(1+\varepsilon+a)(b^2-v^2).$$

Letting $\varepsilon \to 0$, we complete the proof of the lemma. Q.E.D.

LEMMA 3. $\lambda \geq 1/(1+a)\cdot \pi^2/d^2$.

PROOF. By Lemma 2, we have

$$\frac{|\nabla(v/b)|}{\sqrt{1-(v/d)^2}} \leq \lambda^{1/2}(1+a)^{1/2}. \tag{18}$$

Suppose that q_1 and $q_2 \in \overline{\Omega}$ such that $v(q_1) = 1$, $v(q_2) = -1$, and let L be the line segment between q_1 and q_2. L lies on $\overline{\Omega}$ completely, because it is convex. We integrate both sides of (18) along L from q_2 to q_1 and obtain

$$\lambda^{1/2}(1+a)^{1/2}d \geq \lambda^{1/2}(1+a)^{1/2}\text{length of } L \geq \arcsin\frac{1}{b} - \arcsin\frac{-1}{b}. \tag{19}$$

For any $b > 1$, (19) is valid and, letting $b \to 1$, the lemma is completely proved. Q.E.D.

If $a = 0$, then the Theorem is proved. Now suppose $a > 0$. From Lemma 2

$$\frac{|\nabla(v/b)|^2}{1-(v/b)^2} \leq \lambda(1+a), \qquad b > 1.$$

Set $\theta\colon \overline{\Omega} \to R$, $\theta = \arcsin(v/b)$, $\arcsin(-1/b) \leq \theta \leq \arcsin(1/b)$. Then

$$\frac{|\nabla(v/b)|^2}{1-(v/b)^2} = |\nabla\theta|^2 \leq \lambda(1+a).$$

Obviously, $\nabla\theta = 0$ if $v = 0$. Define $F\colon [\arcsin -1/b, \arcsin 1/b] \to R$ as

$$F(\theta_0) = \max_{\substack{x\in\overline{\Omega} \\ \theta(x)=\theta_0}} \frac{|\nabla(v/b)|^2}{1-(v/b)^2}. \tag{20}$$

$F(\theta_0(x))$ is continuous on $\overline{\Omega}$ and

$$F(\theta_0) \le \lambda(1+a), \qquad \theta_0 \in [\arcsin -1/b, \arcsin 1/b].$$

For any $\theta_0 \in [\arcsin -1/b, \arcsin 1/b]$ there must be an $x_0 \in \overline{\Omega}$ such that

$$\theta(x_0) = \theta_0 \quad \text{and} \quad F(\theta_0) = \frac{|\nabla(v/b)|^2}{1-(v/b)^2}(x_0)$$

are valid. Since $a > 0$, we can define a continuous function φ on $\overline{\Omega}$ which satisfies

$$F(\theta) \equiv \lambda\left(1 + \frac{a}{b}\varphi(\theta)\right), \qquad \varphi(\theta) \le b. \tag{21}$$

LEMMA 4. *The C^∞ function $y\colon [\arcsin -1/b, \arcsin 1/b] \to R$ satisfies*
(i) $y(\theta) \ge \varphi(\theta)$, $\theta \in [\arcsin -1/b, \arcsin 1/b]$;
(ii) *there is an $x_0 \in \overline{\Omega}$ such that $\theta(x_0) = \theta_0$ and $y(\theta_0) = \varphi(\theta_0)$;*
(iii) $y(\theta) \ge -1$ *for any* $\theta \in [\arcsin -1/b, \arcsin 1/b]$;
(iv) $y'(\theta_0) \ge 0$.
Then the following inequality is valid:

$$\varphi(\theta_0) \le \sin\theta_0 - y'(\theta_0)\sin\theta_0\cos\theta_0 + \tfrac{1}{2}y''(\theta_0)\cos^2\theta_0.$$

PROOF. Consider the function $\Phi(x)\colon \overline{\Omega} \to R$,

$$\Phi(x) = \left\{\frac{|\nabla v|^2}{b^2 - v^2} - \lambda(1+cy)\right\}\cos^2\theta,$$

where $b > 1$ and $c = a/b$. Obviously, $\Phi(x) \le 0$ for $x \in \overline{\Omega}$ and $\Phi(x_0) = 0$, i.e., $\Phi(x)$ attains its maximum at x_0, since

$$\Phi(x) = \frac{1}{b^2}|\nabla v|^2 - \lambda\left(1 - \frac{v^2}{b^2}\right)(1+cy).$$

If $\nabla v(x_0) = 0$, then

$$0 = \Phi(x_0) = -\lambda(1 - v^2/b^2)(1+cy)|_{x_0}$$

and

$$y(x_0) = -1/c = -a/b < -1.$$

This contradicts the assumption (iii). Thus $\nabla v(x_0) \ne 0$. By Lemma 1, $x_0 \notin \partial\Omega$, i.e., $x_0 \in \Omega$. According to the maximum principle, we have

$$\nabla\Phi(x_0) = 0, \tag{22}$$

$$\Delta\Phi(x_0) \le 0. \tag{23}$$

For convenience we write $\Phi(x)$ as

$$\Phi(x) = \frac{1}{b^2}|\nabla v|^2 - \cos^2\theta(1+cy).$$

Then

$$\Phi_j = \frac{1}{b^2}\sum_i v_i v_{ij} - \lambda(1+cy)(-2\cos\theta\sin\theta)\theta_j - c\lambda\cos^2\theta y'\theta_j.$$

(22) gives that at x_0

$$\frac{2}{b^2}\sum_i v_i v_{ij} = \lambda[cy'\cos^2\theta - 2(1+cy)\cos\theta\sin\theta]\theta_j, \qquad 1\le j\le n. \tag{22$'$}$$

And also

$$\begin{aligned}\Delta\Phi &= \frac{2}{b^2}\sum_{i,j} v_{ij}^2 + \frac{2}{b^2}\nabla v\cdot\nabla(\Delta v) - \lambda c\cos^2\theta\Delta y \\ &\quad - \lambda(1+cy)\Delta\cos^2\theta - 2\lambda c\nabla\cos^2\theta\cdot\nabla y \\ &= \frac{2}{b^2}\sum_{i,j} v_{ij}^2 + \frac{2}{b^2}\nabla v\cdot\nabla(\Delta v) - \lambda c\cos^2\theta(y''|\nabla\theta|^2 + y'\Delta\theta) \\ &\quad + 4\lambda cy'\cos\theta\sin\theta|\nabla\theta|^2 - \lambda(1+cy)\Delta\cos^2\theta.\end{aligned}$$

From (23) we have that at x_0

$$\begin{aligned}0 &\ge \frac{2}{b^2}\sum_{i,j} v_{ij}^2 + \frac{2}{b^2}\nabla v\cdot\nabla(\Delta v) - \lambda c\cos^2\theta(y''|\nabla\theta|^2 + y'\Delta\theta) \\ &\quad + 4\lambda cy'\cos\theta\sin\theta|\nabla\theta|^2 - \lambda(1+cy)\Delta\cos^2\theta.\end{aligned} \tag{23$'$}$$

Since $\nabla v(x_0)\neq 0$, we can choose an orthonormal frame such that $v_1(x_0)\neq 0$ and $v_i(x_0)=0$, $2\le i\le n$. Then by (22$'$) (note $\sin\theta = v/b$, $\theta_j = v_j/b\cos\theta$)

$$\begin{aligned} &v_{i1} = 0, \qquad 2\le i\le n, \\ &v_{11} = (b/2)\lambda[cy'\cos\theta - 2(1+cy)\sin\theta].\end{aligned} \tag{24}$$

Now we compute the terms in (23$'$) at the particular frame

$$\begin{aligned}\nabla v\cdot\nabla(\Delta v)_{x_0} &= \sum_{i,j} v_i v_{jji} = \sum_j v_1(v_{jj})_1 \\ &= v_1\left[-\lambda(v+a) - 2\sum_i v_i(\log f_1)_i\right]_1 \\ &= -\lambda v_1^2 - 2v_1\sum_i v_{i1}(\log f_1)_i - 2v\sum_i v_i(\log f_1)_{i1} \\ &= -\lambda v_1^2 + 2v_1v_{11}(\log f_1)_1 = 2v_1^2(\log f_1)_{11} \quad (\because (24)).\end{aligned} \tag{25}$$

From $\Delta v/b = \Delta\sin\theta = \cos\theta\Delta\theta - \sin\theta|\nabla\theta|^2$, we have

$$\Delta\theta = \frac{1}{\cos\theta}\left[\frac{\Delta v}{b} + \sin\theta|\nabla\theta|^2\right]. \tag{26}$$

And

$$\begin{aligned}\Delta\cos\theta &= \Delta\left(1 - \frac{v^2}{b^2}\right) = -\frac{1}{b^2}\Delta v^2 = -\frac{2}{b^2}(v\Delta v + |\nabla V|^2) \\ &= -\frac{2}{b^2}v\Delta v - 2\cos^2\theta|\nabla\theta|^2.\end{aligned} \tag{27}$$

From $\Phi(x_0) = 0$, we have at x_0

$$(28) \qquad |\nabla\theta|^2 = \frac{1}{b^2}\frac{|\nabla v|^2}{\cos^2\theta} = \frac{1}{b^2}\frac{|\nabla v|^2}{1-(v/b)^2} = \frac{|\nabla v|^2}{b^2 - v^2} = \lambda(1+cy).$$

Putting (25)–(28) into (23′), we obtain that at x_0

$$(29) \qquad \begin{aligned} 0 \ge{} & \frac{2}{b^2}\sum_{i,j} v_{ij}^2 + \frac{2}{b^2}[-\lambda v_1^2 - 2v_1 v_{11}(\log f_1)_1 - 2v_1^2(\log f_1)_{11}] \\ & - \lambda c\left[\lambda y''(1+cy) + y'\left(\frac{\Delta v}{b\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)\lambda(1+cy)\right]\cos^2\theta \\ & + 4(1+cy)\lambda^2 cy'\cos\theta\sin\theta - \lambda(1+cy)\left[-\frac{2}{b^2}v\Delta v - 2\lambda\cos^2\theta(1+cy)\right]. \end{aligned}$$

Since $|\nabla v|^2 = v_1^2 = b^2\cos^2\theta|\nabla\theta|^2 = \lambda^2\cos^2\theta(1+cy)$ ($\because$ (28)) and $\Delta v = -\lambda(v+a) - 2\nabla v\cdot\nabla\log f_1$, (29) can be written as

$$\begin{aligned} 0 \ge{} & \frac{2}{b^2}\sum_{i,j} v_{ij}^2 - 2\lambda^2(1+cy)\cos^2\theta - \lambda^2 cy''(1+cy)\cos^2\theta \\ & + 2\lambda^2 cy'\cos\theta(\sin\theta + c) + 3\lambda^3 cy'(1+cy)\sin\theta\cos\theta \\ & + 2\lambda(1+cy)^2\cos^2\theta - 2\lambda^2(1+cy)\sin\theta(\sin\theta + c) \\ & - \frac{4}{b^2}v_1^2(\log f_1)_{11} - \frac{4}{b^2}v_1 v_{11}(\log f_1)_1 \\ & + \frac{2}{b}\lambda[cy'\cos\theta - 2(1+cy)\sin\theta]v_1(\log f_1)_1. \end{aligned}$$

Putting the second formula of (24) into the above inequality and noting that $(\log f_1)_{11} \le 0$, we have

$$\begin{aligned} 0 \ge{} & \tfrac{1}{2}\lambda^2 c^2(y')^2\cos^2\theta + 2\lambda^2(1+cy)^2 - 2\lambda^2(1+cy)\cos^2\theta \\ & + \lambda^2 cy'[(1+cy)\sin\theta\cos\theta + \cos\theta(\sin\theta + c)] \\ & - \lambda^2 c(1+cy)y''\cos^2\theta - 2\lambda^2(1+cy)\sin\theta(\sin\theta + c). \end{aligned}$$

Dividing both sides of the above inequality by $\lambda^2(1+cy) > 0$, we have

$$0 \ge y'\left(\sin\theta\cos\theta + \cos\frac{\sin\theta + c}{1+c}\right) - y''\cos^2\theta + \frac{2}{c}(1+cy) - \frac{2}{c} - 2\sin\theta,$$

$$2y - 2\sin\theta \le y''\cos^2\theta - y'\left(\sin\theta\cos\theta + \cos\theta\frac{\sin\theta + c}{1+cy}\right).$$

Since $-1 \le y(\theta_0) = \varphi(\theta_0) \le b$, thus,

$$|y(\theta_0)| \le b, \qquad y(\theta_0)\sin\theta_0 \le |y(\theta_0)|\,|\sin\theta_0| \le bv(\theta_0)/b \le 1$$

and

$$c \ge cy\sin\theta, \quad c + \sin\theta \ge (1+cy)\sin\theta, \quad \frac{c+\sin\theta}{1+cy} \ge \sin\theta.$$

Since $y'(\theta_0) \ge 0$, we have

$$\varphi(\theta_0) = y(\theta_0) \le \sin\theta_0 - y'(\theta_0)\sin\theta_0\cos\theta_0 + \tfrac{1}{2}y''(\theta_0)\cos^2\theta_0. \qquad \text{Q.E.D.}$$

LEMMA 5. *Define a function* $\psi\colon [-\pi/2, \pi/2] \to R$ *as*

$$(30)\qquad \begin{cases} \psi(\theta) = \dfrac{(4/\pi)(\theta + \cos\theta\sin\theta) - 2\sin\theta}{\cos^2\theta}, & \theta \in (-\pi/2, \pi/2), \\ \psi(-\pi/2) = -1, \qquad \psi(\pi/2) = 1. \end{cases}$$

Then ψ *is a* C^∞ *function in* $(-\pi/2, \pi/2)$ *and is continuous on* $[-\pi/2, \pi/2]$ *and also* $y = \psi(\theta)$ *satisfies the following equation*:

$$(31)\qquad y - \sin\theta + y'\sin\theta\cos\theta - \tfrac{1}{2}y''\cos^2\theta = 0,$$

and $y'(\theta) \geq 0$, $\theta \in (-\pi/2, \pi/2)$.

PROOF. See reference [**4**].

LEMMA 6. *Let* $\varphi(\theta)$ *be the function defined by* (21). *Then*

$$\varphi(\theta) \leq \psi(\theta), \qquad \theta \in \left[\arcsin\frac{-1}{b}, \arcsin\frac{1}{b}\right],$$

where $\psi(\theta)$ *is defined by* (30).

PROOF. We will use the reduction to absurdity. If

$$(32)\qquad \sigma = \varphi(\theta_0) - \psi(\theta_0) = \max_\theta\{\varphi(\theta) - \psi(\theta)\} > 0$$

we could choose $\psi(\theta) + \sigma = \tilde{y}$ as y in Lemma 4. Therefore,

$$\varphi(\theta_0) = \tilde{y}(\theta_0) = \psi(\theta_0) + \sigma \leq \sin\theta_0 - \psi'(\theta_0)\sin\theta_0\cos\theta_0 + \tfrac{1}{2}\psi''(\theta_0)\cos^2\theta_0 = \psi(\theta_0).$$

This contradicts (32). Q.E.D.

PROOF OF THE THEOREM. By Lemma 6, we have

$$|\nabla\theta|^2 = \frac{|\nabla v|^2}{b^2 - v^2} \leq \lambda\left(1 + \frac{a}{b}\psi(\theta)\right),$$

where $\psi(\theta)$ is the function defined by (30). Hence,

$$(33)\qquad \lambda^{1/2} \geq \frac{|\nabla\theta|}{\sqrt{1 + (a/b)\psi(\theta)}}.$$

Obviously,

$$(34)\qquad \psi(0) = 0, \qquad \psi(-\theta) = -\psi(\theta).$$

Integrating both sides of (33) as in Lemma 3, we obtain

$$\begin{aligned} \lambda^{1/2}d &\geq \int_{\arcsin -1/b}^{\arcsin 1/b} \frac{d\theta}{\sqrt{1 + (a/b)\psi(\theta)}} \\ &= \int_0^{\arcsin 1/b} \left(\frac{1}{\sqrt{1 + (a/b)\psi(\theta)}} + \frac{1}{\sqrt{1 - (a/b)\psi(\theta)}}\right) d\theta. \end{aligned}$$

Since $|\pm(a/b)\psi(\theta)| \leq 1$,

$$\frac{1}{\sqrt{1 + (a/b)\psi(\theta)}} + \frac{1}{\sqrt{1 - (a/b)\psi(\theta)}} = 2\left[1 + \sum_{p-1}^{\infty} \frac{1\cdot 3\cdots(4p-1)}{2\cdot 4\cdots(4p)}\left(\frac{a}{b}\right)^{2p}\psi^{2p}\right] \geq 2.$$

Thus

$$\lambda^{1/2}d \geq 2\arcsin\frac{1}{b}, \qquad \lambda \geq \frac{4}{d^2}\left(\arcsin\frac{1}{b}\right)^2.$$

Letting $b \to 1$, we obtain $\lambda \geq \pi^2/d^2$. Q.E.D.

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